



Numerical solution of nonlinear Volterra–Fredholm integro-differential equations via direct method using triangular functions

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ABSTRACT

An effective direct method to determine the numerical solution of the specific nonlinear Volterra–Fredholm integro-differential equations is proposed. The method is based on new vector forms for representation of triangular functions and its operational matrix. This approach needs no integration, so all calculations can be easily implemented. Some numerical examples are provided to illustrate the accuracy and computational efficiency of the method.

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1. Introduction

Several numerical methods for solving linear and nonlinear integro-differential equations have been presented. Some authors use decomposition method [1,2]. In most methods, a set of basis functions and an appropriate projection method such as Galerkin, collocation, etc. or a direct method have been applied [3–8]. These methods often transform an integro-differential equation to a linear or nonlinear system of algebraic equations which can be solved by direct or iterative methods. In general, generating this system needs calculation of a large number of integrations.

This paper considers a specific case of Volterra–Fredholm integro-differential equations of the form

$$\begin{cases} x'(s) + q(s)x(s) + \lambda_1 \int_0^s k_1(s, t)F(x(t))dt + \lambda_2 \int_0^1 k_2(s, t)G(x(t))dt = y(s), \\ x(0) = x_0, \end{cases} \quad (1)$$

where the functions $F(x(t))$ and $G(x(t))$ are polynomials of $x(t)$ with constant coefficients. For convenience, we put $F(x(t)) = [x(t)]^{n_1}$ and $G(x(t)) = [x(t)]^{n_2}$, where n_1 and n_2 are positive integers. Note that the method presented in this article can be easily extended and applied to any nonlinear integro-differential equations of form (1). It is clear that for $n_1, n_2 = 1$, Eq. (1) is a linear integro-differential equation. Also, without loss of generality, it is supposed that the interval of integration is $[0, 1]$, since any finite interval $[a, b]$ can be transformed to interval $[0, 1]$ by linear maps [5].

For solving these equations, this paper uses a new set of orthogonal functions, introduced by Deb et al. [9]. These functions have been applied for solving variational problems by Babolian et al. [10]. In this article, we present new vector forms of triangular functions (TFs), operational matrix of integration, expansion of functions of one and two variables with respect to triangular functions, and other TFs properties. By using new representations a nonlinear integro-differential equation can be

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easily reduced to a nonlinear system of algebraic equations. The generation of this system needs just sampling of functions, multiplication and addition of matrices and needs no integration.

Finally, we check the proposed method on some examples to show its accuracy and efficiency.

2. Review of triangular functions

Triangular functions have been presented by Deb et al. [9] and studied and used by Babolian et al. [10].

2.1. Definition

Two m -sets of triangular functions (TFs) are defined over the interval $[0, T)$ as

$$\begin{aligned} T1_i(t) &= \begin{cases} 1 - \frac{t - ih}{h}, & ih \leq t < (i + 1)h, \\ 0, & \text{otherwise,} \end{cases} \\ T2_i(t) &= \begin{cases} \frac{t - ih}{h}, & ih \leq t < (i + 1)h, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (2)$$

where $i = 0, 1, \dots, m - 1$, with a positive integer value for m . Also, consider $h = T/m$, and $T1_i$ as the i th left-handed triangular function and $T2_i$ as the i th right-handed triangular function.

In this paper, it is assumed that $T = 1$, so TFs are defined over $[0, 1)$, and $h = 1/m$.

From the definition of TFs, it is clear that triangular functions are disjoint, orthogonal, and complete [9]. Therefore, we can write

$$\int_0^1 T1_i(t)T1_j(t)dt = \int_0^1 T2_i(t)T2_j(t)dt = \begin{cases} \frac{h}{3}, & i = j, \\ 0, & i \neq j. \end{cases} \quad (3)$$

Also,

$$\phi_i(t) = T1_i(t) + T2_i(t), \quad i = 0, 1, \dots, m - 1, \quad (4)$$

where $\phi_i(t)$ is the i th block-pulse function defined as

$$\phi_i(t) = \begin{cases} 1, & ih \leq t < (i + 1)h, \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

where $i = 0, 1, \dots, m - 1$.

2.2. Vector forms

Consider the first m terms of the left-handed triangular functions and the first m terms of the right-handed triangular functions and write them concisely as m -vectors:

$$\begin{aligned} \mathbf{T1}(t) &= [T1_0(t), T1_1(t), \dots, T1_{m-1}(t)]^T, \\ \mathbf{T2}(t) &= [T2_0(t), T2_1(t), \dots, T2_{m-1}(t)]^T, \end{aligned} \quad (6)$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively.

The following properties of the product of two TFs vectors are presented by [10]:

$$\begin{aligned} \mathbf{T1}(t)\mathbf{T1}^T(t) &\simeq \begin{pmatrix} T1_0(t) & 0 & \dots & 0 \\ 0 & T1_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T1_{m-1}(t) \end{pmatrix}, \\ \mathbf{T2}(t)\mathbf{T2}^T(t) &\simeq \begin{pmatrix} T2_0(t) & 0 & \dots & 0 \\ 0 & T2_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & T2_{m-1}(t) \end{pmatrix}, \end{aligned} \quad (7)$$

and

$$\begin{aligned} \mathbf{T1}(t)\mathbf{T2}^T(t) &\simeq \mathbf{0}, \\ \mathbf{T2}(t)\mathbf{T1}^T(t) &\simeq \mathbf{0}, \end{aligned} \quad (8)$$

where $\mathbf{0}$ is the zero $m \times m$ matrix. Also,

$$\begin{aligned}\int_0^1 \mathbf{T1}(t)\mathbf{T1}^T(t)dt &= \int_0^1 \mathbf{T2}(t)\mathbf{T2}^T(t)dt \simeq \frac{h}{3}I, \\ \int_0^1 \mathbf{T1}(t)\mathbf{T2}^T(t)dt &= \int_0^1 \mathbf{T2}(t)\mathbf{T1}^T(t)dt \simeq \frac{h}{6}I,\end{aligned}\quad (9)$$

in which I is an $m \times m$ identity matrix.

2.3. TFs expansion

The expansion of a function $f(t)$ over $[0, 1]$ with respect to TFs, may be compactly written as

$$\begin{aligned}f(t) &\simeq \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) \\ &= \mathbf{c}^T \mathbf{T1}(t) + \mathbf{d}^T \mathbf{T2}(t),\end{aligned}\quad (10)$$

where we may put $c_i = f(ih)$ and $d_i = f((i+1)h)$ for $i = 0, 1, \dots, m-1$. So, approximating a known function by TFs needs no integration to evaluate the coefficients.

2.4. Operational matrix of integration

Expressing $\int_0^s \mathbf{T1}(\tau)d\tau$ and $\int_0^s \mathbf{T2}(\tau)d\tau$ in terms of the triangular functions follows [9]

$$\begin{aligned}\int_0^s \mathbf{T1}(\tau)d\tau &\simeq P1\mathbf{T1}(s) + P2\mathbf{T2}(s), \\ \int_0^s \mathbf{T2}(\tau)d\tau &\simeq P1\mathbf{T1}(s) + P2\mathbf{T2}(s),\end{aligned}\quad (11)$$

where $P1_{m \times m}$ and $P2_{m \times m}$ are called operational matrices of integration in TFs domain and represented as follows:

$$P1 = \frac{h}{2} \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad P2 = \frac{h}{2} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.\quad (12)$$

So, the integral of any function $f(t)$ can be approximated as

$$\begin{aligned}\int_0^s f(\tau)d\tau &\simeq \int_0^s [\mathbf{c}^T \mathbf{T1}(\tau) + \mathbf{d}^T \mathbf{T2}(\tau)] d\tau \\ &\simeq (\mathbf{c} + \mathbf{d})^T P1\mathbf{T1}(s) + (\mathbf{c} + \mathbf{d})^T P2\mathbf{T2}(s).\end{aligned}\quad (13)$$

3. New representation of TFs vector forms and other properties

In this section, we define a new representation of TFs vector forms. Then, some characteristics of TFs are presented using the new definition.

3.1. Definition and expansion

Let $\mathbf{T}(t)$ be a $2m$ -vector defined as

$$\mathbf{T}(t) = \begin{pmatrix} \mathbf{T1}(t) \\ \mathbf{T2}(t) \end{pmatrix}, \quad 0 \leq t < 1,\quad (14)$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ have been defined in (6). Now, the expansion of $f(t)$ with respect to TFs can be written as

$$\begin{aligned}f(t) &\simeq F1^T \mathbf{T1}(t) + F2^T \mathbf{T2}(t) \\ &= F^T \mathbf{T}(t),\end{aligned}\quad (15)$$

where $F1$ and $F2$ are TFs coefficients with $F1_i = f(ih)$ and $F2_i = f((i+1)h)$, for $i = 0, 1, \dots, m-1$. Also, $2m$ -vector F is defined as

$$F = \begin{pmatrix} F1 \\ F2 \end{pmatrix}. \quad (16)$$

Now, assume that $k(s, t)$ is a function of two variables. It can be expanded with respect to TFs as follows:

$$k(s, t) \simeq \mathbf{T}^T(s) K \mathbf{T}(t), \quad (17)$$

where $\mathbf{T}(s)$ and $\mathbf{T}(t)$ are $2m_1$ and $2m_2$ dimensional triangular functions and K is a $2m_1 \times 2m_2$ TFs coefficient matrix. For convenience, we put $m_1 = m_2 = m$. So, matrix K can be written as

$$K = \begin{pmatrix} (K11)_{m \times m} & (K12)_{m \times m} \\ (K21)_{m \times m} & (K22)_{m \times m} \end{pmatrix}, \quad (18)$$

where $K11, K12, K21$, and $K22$ can be computed by sampling the function $k(s, t)$ at points s_i and t_j such that $s_i = ih$ and $t_j = jh$, for $i, j = 0, 1, \dots, m$. Therefore,

$$\begin{aligned} (K11)_{i,j} &= k(s_i, t_j), & i = 0, 1, \dots, m-1, j = 0, 1, \dots, m-1, \\ (K12)_{i,j} &= k(s_i, t_j), & i = 0, 1, \dots, m-1, j = 1, 2, \dots, m, \\ (K21)_{i,j} &= k(s_i, t_j), & i = 1, 2, \dots, m, j = 0, 1, \dots, m-1, \\ (K22)_{i,j} &= k(s_i, t_j), & i = 1, 2, \dots, m, j = 1, 2, \dots, m. \end{aligned} \quad (19)$$

3.2. Product properties

Let X be a $2m$ -vector which can be written as $X^T = (X1^T \ X2^T)$ such that $X1$ and $X2$ are m -vectors. Now, it can be concluded from Eqs. (7) and (8) that

$$\begin{aligned} \mathbf{T}(t) \mathbf{T}^T(t) X &= \begin{pmatrix} \mathbf{T1}(t) \\ \mathbf{T2}(t) \end{pmatrix} \begin{pmatrix} \mathbf{T1}^T(t) & \mathbf{T2}^T(t) \end{pmatrix} \begin{pmatrix} X1 \\ X2 \end{pmatrix} \\ &\simeq \begin{pmatrix} \text{diag}(\mathbf{T1}(t)) & \mathbf{0}_{m \times m} \\ \mathbf{0}_{m \times m} & \text{diag}(\mathbf{T2}(t)) \end{pmatrix} \begin{pmatrix} X1 \\ X2 \end{pmatrix} \\ &= \text{diag}(\mathbf{T}(t)) X \\ &= \text{diag}(X) \mathbf{T}(t). \end{aligned} \quad (20)$$

Therefore,

$$\mathbf{T}(t) \mathbf{T}^T(t) X \simeq \tilde{X} \mathbf{T}(t), \quad (21)$$

where $\tilde{X} = \text{diag}(X)$ is a $2m \times 2m$ diagonal matrix.

Now, let B be a $2m \times 2m$ matrix as

$$B = \begin{pmatrix} (B11)_{m \times m} & (B12)_{m \times m} \\ (B21)_{m \times m} & (B22)_{m \times m} \end{pmatrix}. \quad (22)$$

So, it can be similarly concluded from Eqs. (7) and (8) that

$$\begin{aligned} \mathbf{T}^T(t) B \mathbf{T}(t) &= \begin{pmatrix} \mathbf{T1}^T(t) & \mathbf{T2}^T(t) \end{pmatrix} \begin{pmatrix} B11 & B12 \\ B21 & B22 \end{pmatrix} \begin{pmatrix} \mathbf{T1}(t) \\ \mathbf{T2}(t) \end{pmatrix} \\ &\simeq \mathbf{T1}^T(t) B11 \mathbf{T1}(t) + \mathbf{T2}^T(t) B22 \mathbf{T2}(t) \\ &\simeq \hat{B}11^T \mathbf{T1}(t) + \hat{B}22^T \mathbf{T2}(t), \end{aligned} \quad (23)$$

where $\hat{B}11$ and $\hat{B}22$ are m -vectors with elements equal to the diagonal entries of matrices $B11$ and $B22$, respectively. Therefore,

$$\mathbf{T}^T(t) B \mathbf{T}(t) \simeq \hat{B}^T \mathbf{T}(t), \quad (24)$$

in which \hat{B} is a $2m$ -vector with elements equal to the diagonal entries of matrix B . Also, it is immediately concluded from Eq. (9) that

$$\begin{aligned} \int_0^1 \mathbf{T}(t) \mathbf{T}^T(t) dt &= \int_0^1 \begin{pmatrix} \mathbf{T1}(t) \\ \mathbf{T2}(t) \end{pmatrix} (\mathbf{T1}^T(t) \quad \mathbf{T2}^T(t)) dt \\ &= \int_0^1 \begin{pmatrix} \mathbf{T1}(t) \mathbf{T1}^T(t) & \mathbf{T1}(t) \mathbf{T2}^T(t) \\ \mathbf{T2}(t) \mathbf{T1}^T(t) & \mathbf{T2}(t) \mathbf{T2}^T(t) \end{pmatrix} dt \\ &\simeq \begin{pmatrix} \frac{h}{3} I_{m \times m} & \frac{h}{6} I_{m \times m} \\ \frac{h}{6} I_{m \times m} & \frac{h}{3} I_{m \times m} \end{pmatrix}. \end{aligned} \quad (25)$$

Therefore,

$$\int_0^1 \mathbf{T}(t) \mathbf{T}^T(t) dt \simeq D, \quad (26)$$

where D is the following $2m \times 2m$ matrix:

$$D = \frac{h}{3} \begin{pmatrix} 1 & 0 & \dots & 0 & 1/2 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1/2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1/2 \\ 1/2 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1/2 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/2 & 0 & 0 & \dots & 1 \end{pmatrix}. \quad (27)$$

3.3. Operational matrix

Expressing $\int_0^s \mathbf{T}(\tau) d\tau$ in terms of $\mathbf{T}(s)$, and from Eq. (11), we can write

$$\begin{aligned} \int_0^s \mathbf{T}(\tau) d\tau &= \int_0^s \begin{pmatrix} \mathbf{T1}(\tau) \\ \mathbf{T2}(\tau) \end{pmatrix} d\tau \\ &\simeq \begin{pmatrix} P1 \mathbf{T1}(s) + P2 \mathbf{T2}(s) \\ P1 \mathbf{T1}(s) + P2 \mathbf{T2}(s) \end{pmatrix} \\ &= \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix} \begin{pmatrix} \mathbf{T1}(s) \\ \mathbf{T2}(s) \end{pmatrix}, \end{aligned} \quad (28)$$

so,

$$\int_0^s \mathbf{T}(\tau) d\tau \simeq P \mathbf{T}(s), \quad (29)$$

where $P_{2m \times 2m}$, operational matrix of $\mathbf{T}(s)$, is

$$P = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix}, \quad (30)$$

where $P1$ and $P2$ are given by (12).

Now, the integral of any function $f(t)$ can be approximated as

$$\begin{aligned} \int_0^s f(\tau) d\tau &\simeq \int_0^s F^T \mathbf{T}(\tau) d\tau \\ &\simeq F^T P \mathbf{T}(s). \end{aligned} \quad (31)$$

4. Solving nonlinear integro-differential equation

In this section, using the results obtained in the previous section about triangular functions, an effective and very accurate direct method for solving nonlinear Volterra–Fredholm integro-differential equations is presented.

Consider the following nonlinear Volterra–Fredholm integro-differential equation:

$$\begin{cases} x'(s) + q(s)x(s) + \lambda_1 \int_0^s k_1(s, t)[x(t)]^{n_1} dt + \lambda_2 \int_0^1 k_2(s, t)[x(t)]^{n_2} dt = y(s), \\ x(0) = x_0, \quad 0 \leq s < 1, \quad n_1, n_2 \geq 1, \end{cases} \quad (32)$$

where the parameters λ_1 and λ_2 and \mathcal{L}^2 functions $q(s)$, $y(s)$, $k_1(s, t)$ and $k_2(s, t)$ are known but $x(s)$ is not. Note the appearance of initial condition equation in Eq. (32). This is necessary to ensure the existence of a solution.

Approximating functions $x(s)$, $x'(s)$, $q(s)$, $y(s)$, $[x(s)]^{n_1}$, $[x(s)]^{n_2}$, $k_1(s, t)$ and $k_2(s, t)$ with respect to TFs, (15) and (17) give

$$\begin{aligned} x(s) &\simeq X^T \mathbf{T}(s) = \mathbf{T}^T(s) X, \\ x'(s) &\simeq X'^T \mathbf{T}(s) = \mathbf{T}^T(s) X', \\ q(s) &\simeq Q^T \mathbf{T}(s) = \mathbf{T}^T(s) Q, \\ y(s) &\simeq Y^T \mathbf{T}(s) = \mathbf{T}^T(s) Y, \\ [x(s)]^{n_1} &\simeq X_{n_1}^T \mathbf{T}(s) = \mathbf{T}^T(s) X_{n_1}, \\ [x(s)]^{n_2} &\simeq X_{n_2}^T \mathbf{T}(s) = \mathbf{T}^T(s) X_{n_2}, \\ k_1(s, t) &\simeq \mathbf{T}^T(s) K_1 \mathbf{T}(t), \\ k_2(s, t) &\simeq \mathbf{T}^T(s) K_2 \mathbf{T}(t), \end{aligned} \quad (33)$$

where $2m$ -vectors X , X' , Q , Y , X_{n_1} , X_{n_2} and $2m \times 2m$ matrices K_1 and K_2 are TFs coefficients of $x(s)$, $x'(s)$, $q(s)$, $y(s)$, $[x(s)]^{n_1}$, $[x(s)]^{n_2}$, $k_1(s, t)$ and $k_2(s, t)$, respectively.

Now, we require the following lemma:

Lemma 1. Let $2m$ -vectors X and X_n be TFs coefficients of $x(s)$ and $[x(s)]^n$, respectively. If

$$X = (X_1^T \quad X_2^T)^T = (X_{10}, X_{11}, \dots, X_{1m-1}, X_{20}, X_{21}, \dots, X_{2m-1})^T, \quad (34)$$

then

$$X_n = (X_{10}^n, X_{11}^n, \dots, X_{1m-1}^n, X_{20}^n, X_{21}^n, \dots, X_{2m-1}^n)^T, \quad (35)$$

where $n \geq 1$, is a positive integer.

Proof. When $n = 1$, (35) follows at once from $[x(s)]^n = x(s)$. Suppose that (35) holds for n , we shall deduce it for $n + 1$.

Since $[x(s)]^{n+1} = x(s)[x(s)]^n$, from (33) and (21) it follows that

$$\begin{aligned} [x(s)]^{n+1} &\simeq (X^T \mathbf{T}(s)) \cdot (X_n^T \mathbf{T}(s)) \\ &= X^T \mathbf{T}(s) \mathbf{T}^T(s) X_n \\ &\simeq X^T \tilde{X}_n \mathbf{T}(s). \end{aligned} \quad (36)$$

Now, using (35) we obtain

$$X^T \tilde{X}_n = (X_{10}^{n+1}, X_{11}^{n+1}, \dots, X_{1m-1}^{n+1}, X_{20}^{n+1}, X_{21}^{n+1}, \dots, X_{2m-1}^{n+1})^T, \quad (37)$$

therefore (35) holds for $n + 1$, and the lemma is established. \square

So, the components of X_n can be computed in terms of components of unknown vector X .

For solving Eq. (32), we substitute (33) into (32), therefore,

$$Y^T \mathbf{T}(s) \simeq X'^T \mathbf{T}(s) + Q^T \mathbf{T}(s) \mathbf{T}^T(s) X + \lambda_1 \mathbf{T}^T(s) K_1 \int_0^s \mathbf{T}(t) \mathbf{T}^T(t) X_{n_1} dt + \lambda_2 \mathbf{T}^T(s) K_2 \int_0^1 \mathbf{T}(t) \mathbf{T}^T(t) X_{n_2} dt. \quad (38)$$

Using Eqs. (21) and (26) it follows that

$$Y^T \mathbf{T}(s) \simeq X'^T \mathbf{T}(s) + (\tilde{Q} \mathbf{T}(s))^T X + \lambda_1 \mathbf{T}^T(s) K_1 \tilde{X}_{n_1} \int_0^s \mathbf{T}(t) dt + \lambda_2 \mathbf{T}^T(s) K_2 D X_{n_2}. \quad (39)$$

Using operational matrix P , in Eq. (30), gives

$$Y^T \mathbf{T}(s) \simeq X'^T \mathbf{T}(s) + X^T \tilde{Q} \mathbf{T}(s) + \lambda_1 \mathbf{T}^T(s) K_1 \tilde{X}_{n_1} P \mathbf{T}(s) + \lambda_2 (K_2 D X_{n_2})^T \mathbf{T}(s), \quad (40)$$

in which $\lambda_1 K_1 \tilde{X}_{n_1} P$ is a $2m \times 2m$ matrix. Eq. (24) follows

$$\mathbf{T}^T(s) \lambda_1 K_1 \tilde{X}_{n_1} P \mathbf{T}(s) \simeq \hat{X}_{n_1}^T \mathbf{T}(s), \quad (41)$$

where \hat{X}_{n_1} is a $2m$ -vector with components equal to the diagonal entries of the matrix $\lambda_1 K_1 \tilde{X}_{n_1} P$. Combining (40) and (41) gives

$$Y^T \mathbf{T}(s) \simeq X'^T \mathbf{T}(s) + X^T \tilde{Q} \mathbf{T}(s) + \hat{X}_{n_1}^T \mathbf{T}(s) + \lambda_2 (K_2 D X_{n_2})^T \mathbf{T}(s), \quad (42)$$

or

$$X' + \tilde{Q}X + \hat{X}_{n_1} + \lambda_2 K_2 D X_{n_2} \simeq Y. \quad (43)$$

Note that \tilde{Q} is a diagonal matrix, so $\tilde{Q}^T = \tilde{Q}$.

Now, X' must be computed in terms of X . Note that

$$\begin{aligned} x(s) - x(0) &= \int_0^s x'(\tau) d\tau \\ &\simeq \int_0^s X'^T \mathbf{T}(\tau) d\tau \\ &\simeq X'^T P \mathbf{T}(s). \end{aligned} \quad (44)$$

Therefore,

$$x(s) \simeq X'^T P \mathbf{T}(s) + X_0^T \mathbf{T}(s), \quad (45)$$

where X_0 is the $2m$ -vector of the form $X_0 = [x_0, x_0, \dots, x_0]^T$, consequently, using (33) gives

$$X \simeq P^T X' + X_0. \quad (46)$$

Now, combining (43) and (46) and replacing \simeq with $=$, it follows that

$$(I + P^T \tilde{Q})X + P^T \hat{X}_{n_1} + \lambda_2 P^T K_2 D X_{n_2} = P^T Y + X_0. \quad (47)$$

Eq. (47) is a nonlinear system of $2m$ algebraic equations for the $2m$ unknowns $X_{10}, X_{11}, \dots, X_{1m-1}, X_{20}, X_{21}, \dots, X_{2m-1}$. Components of $X^T = (X_1^T \ X_2^T)$ can be obtained by an iterative method. Hence, an approximate solution $x(s) \simeq X^T \mathbf{T}(s)$, or $x(s) \simeq X_1^T \mathbf{T}_1(s) + X_2^T \mathbf{T}_2(s)$ can be computed for Eq. (32) without using any projection method.

5. Numerical examples

Now, the direct method presented in this article is checked on four examples. Three examples are selected from different references, so their numerical results obtained here can be compared with the exact solutions, the approximate solutions using the BPFs, and those of other numerical methods.

The computations associated with the examples were performed using Matlab 7 on a Personal Computer.

Example 1. Consider the following nonlinear integro-differential equation:

$$x'(s) + x(s) + \frac{1}{2} \int_0^s s x^2(t) dt - \frac{1}{4} \int_0^1 t x^3(t) dt = y(s), \quad (48)$$

where $y(s) = 2s + s^2 + \frac{1}{10}s^6 - \frac{1}{32}$, with the initial condition $x(0) = 0$, and the exact solution $x(s) = s^2$. The numerical results are shown in Table 1.

Example 2. For the following nonlinear integro-differential equation [6,7]:

$$x'(s) + 2sx(s) - \int_0^s (s+t)x^3(t) dt - \int_0^1 (s-t)x(t) dt = y(s), \quad (49)$$

where $y(s) = (-\frac{2}{3}s + \frac{1}{9})e^{3s} + (2s+1)e^s + (\frac{4}{3}-e)s + \frac{8}{9}$, with the initial condition $x(0) = 1$, and the exact solution $x(s) = e^s$, Table 2 presents a comparison of the numerical solutions with the presented solution.

Table 1

Numerical results for Example 1.

s	Exact solution	Presented method ($m = 16$)	Presented method ($m = 32$)	BPFs method ($m = 32$)
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.010000	0.010978	0.010166	0.012184
0.2	0.040000	0.040702	0.040254	0.041463
0.3	0.090000	0.090736	0.090262	0.088322
0.4	0.160000	0.161077	0.160191	0.152761
0.5	0.250000	0.250164	0.250041	0.266024
0.6	0.360000	0.361120	0.360202	0.371480
0.7	0.490000	0.490819	0.490283	0.494515
0.8	0.640000	0.640819	0.640283	0.635128
0.9	0.810000	0.811118	0.810202	0.793318
1.0	1.000000	1.000149	1.000037	0.969086

Table 2

Numerical results for Example 2.

s	Exact solution	Presented method ($m = 16$)	Presented method ($m = 32$)	BPFs method ($m = 32$)	Taylor polynomial method ($N = 7$)
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	1.105171	1.105548	1.105223	1.115708	–
0.2	1.221403	1.221578	1.221494	1.225369	1.221403
0.3	1.349859	1.350088	1.349971	1.345821	–
0.4	1.491825	1.492494	1.491933	1.478128	1.491825
0.5	1.648721	1.649007	1.648795	1.675019	–
0.6	1.822119	1.823866	1.822484	1.839752	1.822120
0.7	2.013753	2.016314	2.014465	2.020740	–
0.8	2.225541	2.229950	2.226719	2.219630	2.225542
0.9	2.459603	2.467818	2.461507	2.438278	–
1.0	2.718282	2.731444	2.721505	2.678832	2.718281

Table 3

Numerical results for Example 3.

s	Exact solution	Presented method ($m = 8$)	Presented method ($m = 16$)	BPFs method ($m = 16$)	Adomian's method
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.995004	0.993837	0.994555	0.995141	0.994951
0.2	0.980067	0.978496	0.979825	0.975784	0.980303
0.3	0.955336	0.954105	0.955174	0.960386	0.955685
0.4	0.921061	0.920841	0.920861	0.918443	0.921165
0.5	0.877583	0.878921	0.877921	0.862193	0.877048
0.6	0.825336	0.826070	0.825397	0.828963	0.822596
0.7	0.764842	0.765608	0.765164	0.752929	0.755333
0.8	0.696707	0.698009	0.697142	0.710418	0.667739
0.9	0.621610	0.623786	0.622057	0.617232	0.547241
1.0	0.540302	0.543491	0.541102	0.566917	0.364798

Example 3. For the following nonlinear integro-differential equation [1,7]:

$$x'(s) + \int_0^s 3 \cos(s-t) x^2(t) dt = y(s), \quad (50)$$

where $y(s) = 2 \sin s \cos s$, with the initial condition $x(0) = 1$, and the exact solution $x(s) = \cos s$, Table 3 shows the numerical results and comparison with the other numerical solutions.

Example 4. Consider the following nonlinear Volterra integro-differential equation [4,8]:

$$x'(s) + sx(s) - \int_0^s x^2(t) dt = -1, \quad (51)$$

with the initial condition $x(0) = 0$. The exact solution of this example can be found analytically by reducing to differential equation, but the analytical solution is not represented by the elementary functions. The numerical results are shown in Table 4.

6. Comment on the results

The direct method based on the TFs and their operational matrix transforms a nonlinear Volterra–Fredholm integro-differential equation to a set of algebraic equations without applying any projection method. Solving this system by an

Table 4
Numerical results for Example 4.

s	Exact solution	Presented method ($m = 64$)	BPFs method ($m = 64$)	Walsh series method ($m = 60$)	Chebyshev polynomial method ($N = 6$)
0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
0.0625	−0.06250	−0.06250	−0.07031	−0.06250	−0.06250
0.1250	−0.12498	−0.12498	−0.13279	−0.12498	−0.12498
0.1875	−0.18740	−0.18740	−0.19519	−0.18740	−0.18740
0.2500	−0.24967	−0.24967	−0.25744	−0.24967	−0.24967
0.3125	−0.31171	−0.31170	−0.31943	−0.31171	−0.31171
0.3750	−0.37336	−0.37335	−0.38102	−0.37336	−0.37336
0.4375	−0.43446	−0.43445	−0.44204	−0.43446	−0.43446
0.5000	−0.49482	−0.49481	−0.50230	−0.49482	−0.49482
0.5625	−0.55423	−0.55422	−0.56156	−0.55423	−0.55423
0.6250	−0.61243	−0.61242	−0.61959	−0.61243	−0.61243
0.6875	−0.66917	−0.66915	−0.67612	−0.66916	−0.66917
0.7500	−0.72415	−0.72413	−0.73087	−0.72415	−0.72415
0.8125	−0.77709	−0.77707	−0.78352	−0.77709	−0.77709
0.8750	−0.82767	−0.82764	−0.83378	−0.82766	−0.82767
0.9375	−0.87557	−0.87554	−0.88132	−0.87557	−0.87557
1.0000	−0.92048	−0.92044	−0.91501	−0.92047	−0.92047

iterative method gives an approximate solution $x(s) \simeq X^T \mathbf{T}(s)$ or $x(s) \simeq X1^T \mathbf{T1}(s) + X2^T \mathbf{T2}(s)$ for Eq. (32). It is clear that $x(s)$ is a linear combination of $2m$ piecewise linear orthogonal triangular functions.

Its applicability and accuracy were checked on four examples. In these examples the approximate solutions are briefly compared with the exact and approximate solutions obtained by the method proposed in [7] which is implemented using the BPFs. It follows from the numerical results that the accuracy of the solutions obtained using the TFs is quite good in comparison with the BPFs. Also, increasing the number of TFs over $[0, 1)$ decreases the error of the solution rapidly. To show its convergence and stability, the current method can be run with increasing m , until the computed results have appropriate accuracy.

For Examples 2–4, the numerical results presented in Tables 2–4 are briefly compared with the approximate solutions obtained by the other methods. Although this comparison does not show the categorical superiority of the proposed method over the others from the viewpoint of accuracy, it seems that the number of calculations in the direct method is considerably less than that of the other methods. This is due to the fact that the generation of the algebraic equations' system in the current method needs just sampling of functions, multiplication and addition of matrices, and needs no integration. So, this method may be run very quickly even for large values of m .

The main advantage of this method is low cost of setting up the equations without using any projection method and any integration.

Finally, the method can be easily extended and applied to nonlinear Volterra and Fredholm integro-differential equations of arbitrary order and systems of integro-differential equations with suitable initial conditions.

References

- [1] K. Maleknejad, M. Hadizadeh, The numerical analysis of Adomian's decomposition method for nonlinear Volterra integral and integro-differential equations, *International Journal of Engineering Science*, Iran University of Science & Technology 8 (2a) (1997) 33–48.
- [2] A.M. Wazwaz, *A First Course in Integral Equations*, World Scientific, Singapore, 1997.
- [3] H. Brunner, *Collocation Method for Volterra Integral and Relation Functional Equations*, Cambridge University Press, Cambridge, 2004.
- [4] H. Çerdik-Yaslan, A. Akyüz-Daşcıoğlu, Chebyshev polynomial solution of nonlinear Fredholm-Volterra integro-differential equations, *Journal of Arts and Sciences*, Cankaya University 5 (2006) 89–101.
- [5] L.M. Delves, J.L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge, 1985.
- [6] K. Maleknejad, Y. Mahmoudi, Taylor polynomial solution of high-order nonlinear Volterra–Fredholm integro-differential equations, *Applied Mathematics and Computation* 145 (2003) 641–653.
- [7] E. Babolian, Z. Masouri, S. Hatamzadeh-Varmazyar, New direct method to solve nonlinear Volterra–Fredholm integral and integro differential equation using operational matrix with Block-pulse functions, *Progress In Electromagnetics Research B* 8 (2008) 59–76.
- [8] B. Sepehrian, M. Razzaghi, Single-term Walsh series method for the Volterra integro-differential equations, *Engineering Analysis with Boundary Elements* 28 (2004) 1315–1319.
- [9] A. Deb, A. Dasgupta, G. Sarkar, A new set of orthogonal functions and its application to the analysis of dynamic systems, *Journal of the Franklin Institute* 343 (2006) 1–26.
- [10] E. Babolian, R. Mokhtari, M. Salmani, Using direct method for solving variational problems via triangular orthogonal functions, *Applied Mathematics and Computation* 191 (2007) 206–217.